Healthiness from Duality

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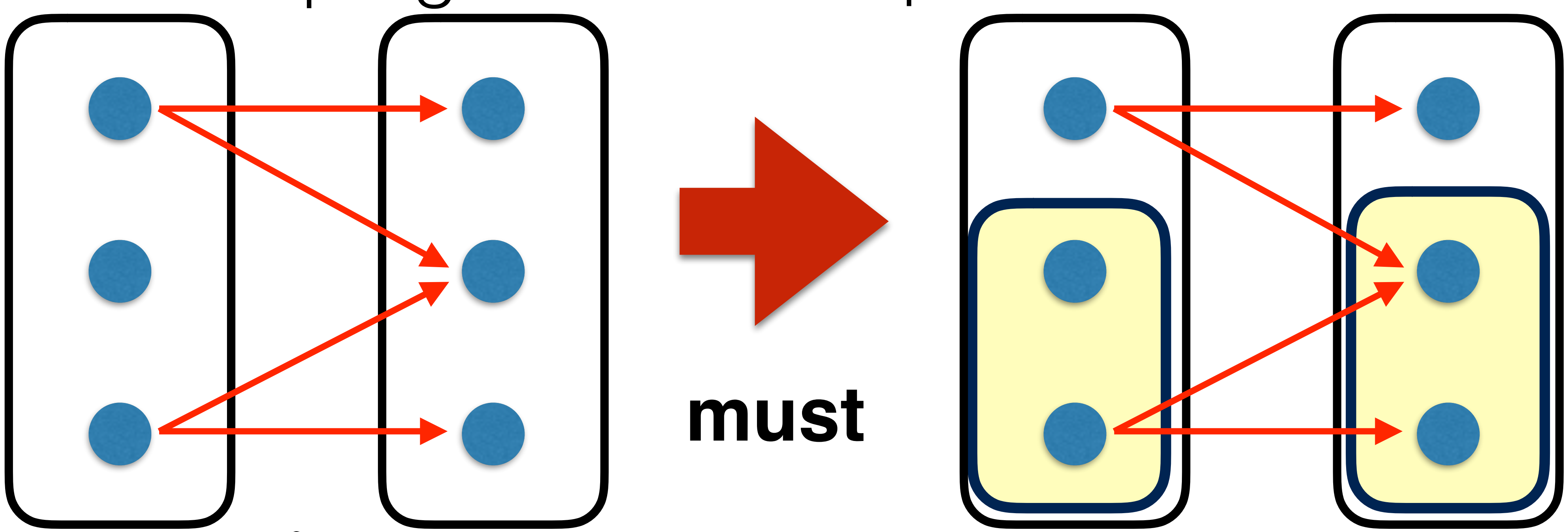
Today’s goal

- **Categorical** predicate transformer semantics
  - unifying [Hasuo 2014] and [Jacobs CALCO 2015] with **relative algebra**
  - enabling formulation of **healthiness condition**
- Extension to the alternating cases

Predicate Transformer Semantics

Interpreting a computation (= a Kleisli arrow)
as a **backward predicate transformer**

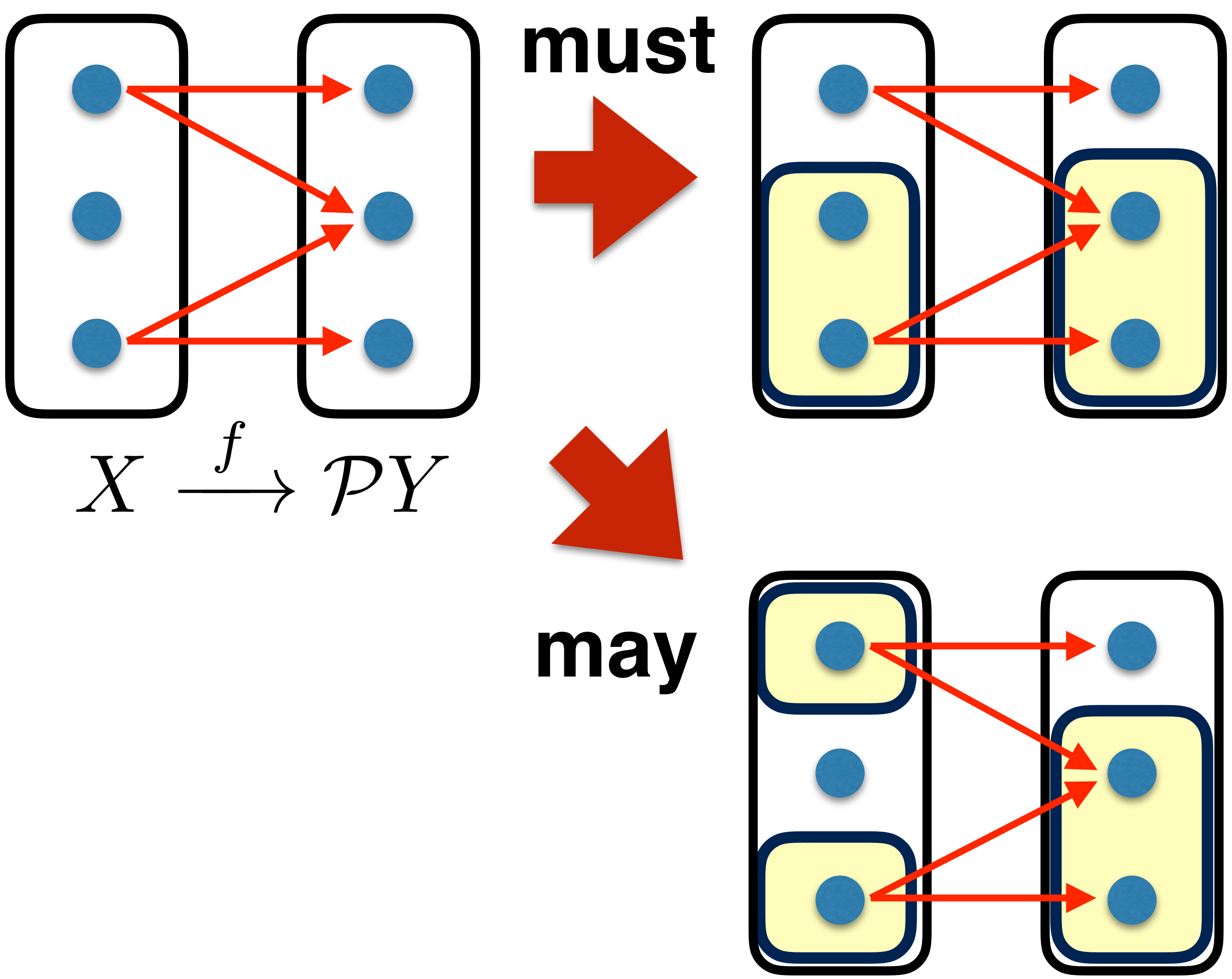
nondet. program predicate transformer



$X \xrightarrow{f} \mathcal{P}Y$ $\qquad$ $2^X \longleftarrow 2^Y$

$$\{x \in X \mid \forall y \in f(x).\ y \in S\} \longleftarrow\!\!\dashv S \subseteq Y$$

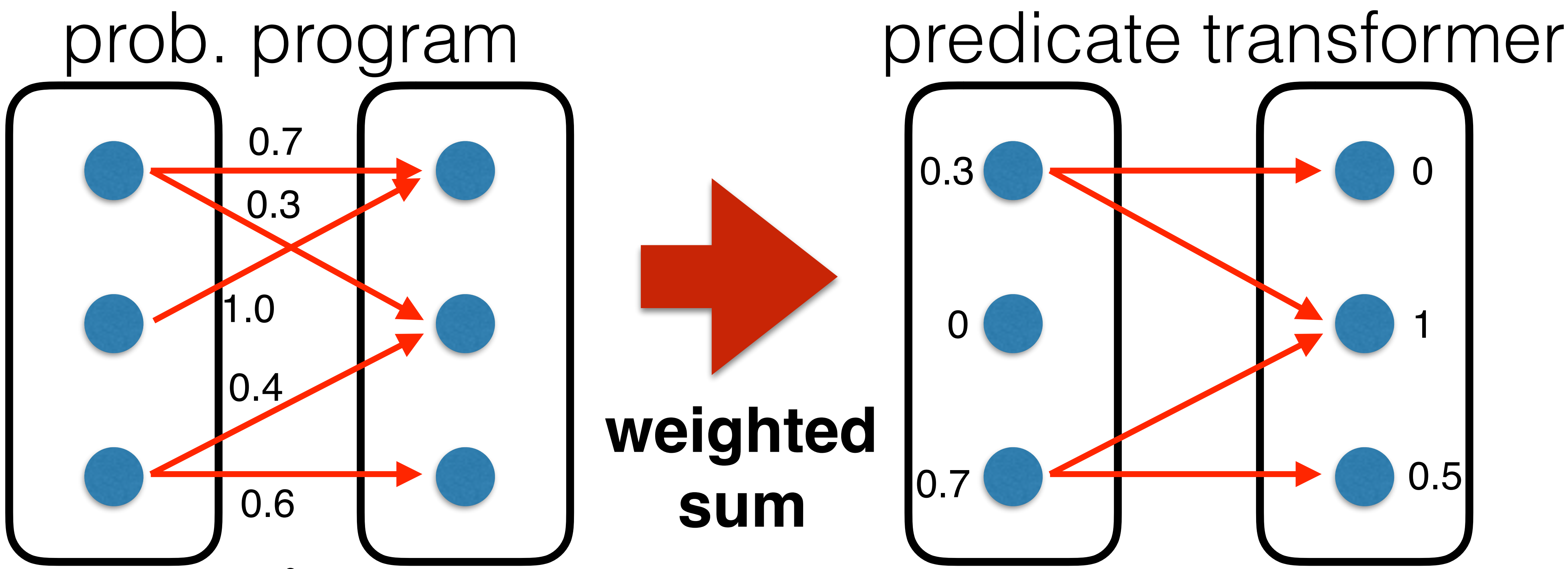
Remark: There might be multiple choices of PT semantics for a single type of branching.



$$\{x \in X \mid \forall y \in f(x).\ y \in S\} \longleftarrow\!\shortmid\ S \subseteq Y$$

$$\{x \in X \mid \exists y \in f(x).\ y \in S\} \longleftarrow\!\shortmid\ S \subseteq Y$$

Probabilistic Example



$$X \xrightarrow{f} \mathcal{D}Y \qquad [0,1]^X \longleftarrow [0,1]^Y$$

$$\left\{ \begin{array}{l} \text{probability} \\ \text{distributions on } Y \end{array} \right\}$$

$$(\lambda x. \sum_{y \in Y} p(y) f(x,y)) \mapsfrom (p\colon Y \to [0,1])$$

Healthiness condition

Healthiness condition: what kind of predicate transformer comes from a Kleisli arrow?

e.g.

$$\boxed{\begin{array}{l}\textbf{Thm.}\ \text{for } \varphi\colon 2^Y \to 2^X, \\ \varphi = \mathbb{P}^{\Diamond}(f) \text{ for some } f\colon X \to \mathcal{P}Y \iff \varphi \text{ is join-preserving}\end{array}}$$

In other words,

$$\begin{array}{l}\mathbb{P}^{\Diamond}\colon \mathcal{K}\ell(\mathcal{P}) \longrightarrow \mathbf{CL}_{\bigvee}^{\mathrm{op}} \\ (X \to PY) \mapsto (2^X \leftarrow 2^Y)\end{array}$$

is well-defined & full.

Categorical understandings of PT semantics

Recipe 1: adjunction recipe

[Jacobs CALCO 2015]

Observation: we have a decomposition:

$$\boxed{\mathcal{P}\cong[2^{(-)},2]_{\bigvee}} \;\circlearrowright\; \begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "2^{(-)}"] \arrow[r, phantom, "\perp"] & \mathbf{CL}_{\bigvee}^{\mathrm{op}} \arrow[l, bend left, "{[-,2]_{\bigvee}}"] \end{tikzcd}$$

$$\boxed{\begin{aligned} \mathcal{P}X &\longrightarrow [2^X, 2]_{\bigvee} \\ S &\mapsto \Big(\chi \mapsto \bigvee\nolimits_{x\in S} \chi(x)\Big) \end{aligned}}$$

then we have

$$
\begin{tikzcd}
(X \mapsto \mathcal{P}Y) \arrow[rrr, mapsto, "\mathbb{P}^{\Diamond}"] & & & (2^{Y} \to 2^{X}) \\
\mathcal{K}\ell(\mathcal{P}) \arrow[r, "\mathcal{K}\ell(\sigma)", "\cong"'] & \mathcal{K}\ell\big([2^{(-)}, 2]_{\bigvee}\big) \arrow[rr, "K"] \arrow[dr, bend left=15] & & (\mathbf{CL}_{\bigvee})^{\mathrm{op}} \arrow[dl, bend left=15] \\
& & \mathbf{Set} \arrow[ul, bend left=15, "\dashv"] \arrow[ur, bend left=15, "\dashv"'] &
\end{tikzcd}
$$

and the resulting functor $\mathcal{K}\ell(\mathcal{P}) \longrightarrow (\mathbf{CL}_{\bigvee})^{\mathrm{op}}$
is **fully faithful** (since so is comparison functor $\boldsymbol{K}$),

→ healthiness condition!

Summary of adjunction recipe

Key: decomposing a monad into a dual adjunction

✓ healthiness condition for free
✗ decomposition is hard to find
✗ hiding the use of modality (e.g. may vs. must)

Recipe 2: modality recipe

Observation: modality = Eilenberg-Moore algebra
[Moggi 1991, Hasuo 2014]

e.g. May-modality (for powerset)

$= \mathcal{P}2 \xrightarrow{\bigvee} 2$ (join-semilattice structure)

Using this “modality”, we can define PT semantics as

$$\mathbb{P}^{\Diamond} : \mathcal{K}\ell(\mathcal{P}) \longrightarrow \mathbf{Set}^{\mathrm{op}}$$

$$(X \xrightarrow{f} PY) \mapsto \underline{(2^X \leftarrow 2^Y)}$$

$$\boxed{\dfrac{Y \xrightarrow{\chi} 2}{\begin{array}{ccc} \mathcal{P}Y & \xrightarrow{\mathcal{P}\chi} & \mathcal{P}2 \\ f\uparrow & & \downarrow\bigvee \\ X & \dashrightarrow & 2 \end{array}}}$$

Summary of modality recipe

Key: Use of modality = EM-algebra over truth values

✓ concrete description of PT semantics

✓ able to distinguish “must vs. may”
(as choice of $\mathcal{P}2 \xrightarrow{\bigvee} 2$ or $\mathcal{P}2 \xrightarrow{\bigwedge} 2$)

✗ domain of interpretation is restricted to $\mathbf{Set}$

✗ too loose to acquire healthiness result

Problem

- How to **unify** these 2 approaches?
  - **adjunction** recipe & **modality** recipe
- we want to get
  - precise healthiness result
  - concrete description of semantics by modality

Key observation

A modality $T\Omega \xrightarrow{\tau} \Omega$ defines Set-valued semantics

$$\begin{aligned} \mathcal{K}\ell(T) &\longrightarrow \mathbf{Set}^{\mathrm{op}} \\ X &\longmapsto \Omega^X \end{aligned}$$

since Ω is a set.

X-fold product of $\mathbf{\Omega}$

To acquire

$$Kl(\mathcal{P}) \longrightarrow \mathcal{D}^{\mathrm{op}}$$

we want $\boldsymbol{T}$-algebra whose underlying space is in $\mathcal{D}$.

$\Rightarrow$ How to formalize it?

Equivalent formulation of Eilenberg-Moore algebra

Thm (Kelly?). T: monad on $\mathbf{Set}$, A: set, then

$$\frac{\alpha \colon TA \to A \colon \text{EM-alg}}{\alpha^\sharp \colon T \to \underline{\mathbf{Set}(A^{(-)}, A)} \colon \text{monad map}}$$

$$\begin{tikzcd}
\mathbf{Set}(A^{(-)},A) \arrow[r, phantom] & \mathbf{Set} \arrow[loop left] \arrow[r, bend left, "A^{(-)}"] \arrow[r, phantom, "\perp"] & \mathbf{Set}^{\mathrm{op}} \arrow[l, bend left, "A^{(-)}"]
\end{tikzcd}$$

Universal algebraic perspective

The monad map can be understood as:

$$\alpha^\sharp_X : TX \longrightarrow \mathbf{Set}(A^X, A)$$
$$t \longmapsto \alpha^\sharp_X(t)$$

term over X

its interpretation

Relative algebra

Def. ($\mathcal{D}$: complete cat.)
a $\mathcal{D}$-relative T-algebra is a pair $(A, \bar{\alpha})$ where

- $A \in \mathcal{D}$
- $\bar{\alpha} : T \to \underline{\mathcal{D}(A^{(-)}, A)} :$ monad map

$$\boxed{\begin{tikzcd} \mathcal{D}(A^{(-)},A) \; \mathbf{Set} \arrow[loop left] \arrow[rr, bend left, "A^{(-)}"] \arrow[rr, phantom, "\perp"] & & \mathcal{D}^{\mathrm{op}} \arrow[ll, bend left, "\mathcal{D}(-,A)"] \end{tikzcd}}$$

Relative algebra recipe

Ingredient: $\mathcal{D}$: complete category (for predicates)
(Ω, τ): relative $\boldsymbol{T}$-algbera (modality)

$\tau : T \to \mathcal{D}(\Omega^{(-)}, \Omega)$

then we can define PT-semantics as

$$\begin{array}{c} \mathcal{K}\ell(T) \\ \Big\downarrow {\scriptstyle \mathbb{P}^{\tau}} \\ \mathcal{D}^{\mathrm{op}} \end{array} \qquad \dfrac{X \longrightarrow TY \text{ in } \mathbf{Set}}{\dfrac{X \to TY \to \mathcal{D}(\Omega^{Y}, \Omega)}{\Omega^{Y} \longrightarrow \Omega^{X} \text{ in } \mathcal{D}}}$$

Healthiness result

Recall we have $\mathbb{P}^\tau : \mathcal{K}\ell(T) \to \mathcal{D}^{\mathrm{op}}$ for relative T-alg (Ω, τ).

Thm (Healthiness result).
$\mathbb{P}^\tau_{X,Y} : \mathcal{K}\ell(T)(X, Y) \to \mathcal{D}(\Omega^Y, \Omega^X)$ is surjective (injective) if $\tau_Y : TY \to \mathcal{D}(\Omega^Y, \Omega)$ is surjective (injective)

Problem of relative algebra

$$(A\colon \mathcal{D}\text{-object},\ \tau\colon \underline{T \to \mathcal{D}(A^{(-)}, A)})$$

- Too abstract, difficult to construct explicitly
  - to define a relative algebra, we need a natural transformation = large amount of data
- cf.) a T-algebra = an object & a morphism

Construct a relative algebra

Assume $U_{\mathcal{D}} \colon \mathcal{D} \to \mathbf{Set}$: faithful and continuous

Thm. there is a bijective correspondence

$$\frac{\text{a } \mathcal{D}\text{-relative } T\text{-algebra}}{A_{\mathcal{D}} \in \mathcal{D} \ \text{ and } \ TA \xrightarrow{a} A \ \text{ subject to}}$$

- $U_{\mathcal{D}}(A_{\mathcal{D}}) = A$
- satisfies the following lifting condition:

$$\begin{tikzcd}
 & \mathcal{D}(A_{\mathcal{D}}^{X}, A_{\mathcal{D}}) \arrow[d, "U_{\mathcal{D}}"] \\
TX \arrow[ur, dashed, "\exists \overline{a^{\sharp}}_{X}"] \arrow[r, "a^{\sharp}_{X}"'] & \mathbf{Set}(A^{X}, A)
\end{tikzcd}$$

Examples

- $\tau_\Diamond \colon \mathcal{P}2 \xrightarrow{\bigvee} 2$ induces $\mathbf{CL}_\bigvee$-relative algebra, with $\tau_\Diamond^\sharp \colon \mathcal{P} \to [2^{(-)}, 2]_\bigvee$ an isomorphism. $\longrightarrow \mathbb{P}^{\tau_\Diamond} = \mathbb{P}^\Diamond$ is fully faithful.

- $\tau = \int \colon \mathcal{D}[0,1] \to [0,1]$ induces $\mathbf{EMod}$-relative algebra, and $\tau^\sharp \colon \mathcal{D}X \to \mathbf{EMod}([0,1]^X, [0,1])$ is bijective when X finite. $\longrightarrow$ healthiness for finite states holds.

(These results are already known)

Summary of relative algebra recipe

- introduced new categorical formulation of PT semantics, unifying our two works

  ✓ fine enough to explain helathiness condition from categorical point of view

  ✓concrete description using “modality”

Missing Link

- We have defined a functor:

$$\mathbb{P}^{\bar{\tau}} : \mathcal{K}\ell(T) \longrightarrow \mathcal{D}^{\mathrm{op}}$$

$$(X \to TY) \mapsto (\Omega^X \leftarrow \Omega^Y)$$

- It only involves **Kleisli category**,
  with **Eilenberg-Moore category** missing.

In fact, it is a part of larger picture (if $\mathcal{D}$ is complete)

$$
\begin{tikzcd}
\mathcal{D}^{\mathrm{op}} \arrow[rr, bend left, "{[-,\overline{\Omega}]_{\mathcal{D}}}"] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(T) \arrow[ll, bend left, "{[-,\overline{\Omega}]_{T}}"] \\
& \mathcal{K}\ell(T) \arrow[ul, "\mathbb{P}^{T}"] \arrow[ur, "K"'] &
\end{tikzcd}
$$

State-and-effect triangle

[Jacobs]

More details

Setting: $\mathcal{D}$: complete and concrete,
$\overline{\Omega} = (\Omega, \tau)$: $\mathcal{D}$-relative T-algebra

then we have

- We have dual adjunction
$$\begin{tikzcd} \mathcal{D}^{\mathrm{op}} \arrow[r, bend left, "{[-,\overline{\Omega}]_{\mathcal{D}}}"] \arrow[r, phantom, "\top"] & \mathcal{EM}(T) \arrow[l, bend left, "{[-,\overline{\Omega}]_{T}}"] \end{tikzcd}$$
    - “over” Hom-functors (into “$\overline{\Omega}$”)

$$\begin{tikzcd} \mathcal{D}^{\mathrm{op}} \arrow[rr, bend left, "{[-,\overline{\Omega}]_{\mathcal{D}}}"] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(T) \arrow[ll, bend left, "{[-,\overline{\Omega}]_{T}}"] \\ & \mathcal{K}\ell(T) \arrow[ul, "\mathbb{P}^{\tau}"] \arrow[ur, "K"'] & \end{tikzcd}$$

- Factors through $\mathbb{P}^{\bar{\tau}} : \mathcal{K}\ell(\mathcal{P}) \longrightarrow \mathcal{D}^{\mathrm{op}}$
i.e. the following commutes.

$$\begin{tikzcd} \mathcal{D}^{\mathrm{op}} \arrow[rr, "{[-,\overline{\Omega}]_{\mathcal{D}}}"] & & \mathcal{EM}(T) \\ & \mathcal{K}\ell(T) \arrow[ul, "\mathbb{P}^{\tau}"] \arrow[ur, "K"'] & \end{tikzcd}$$

Why is this important?

We will use it for **alternating branching** case!

Alternating branching

- mixing 2 types of branching
  - nondet. & nondet. (player vs. opponent)
  - nondet. & prob. (opponent vs. environment) [Morgan, McIver, Seidel 1996]
- formulated in [Hasuo 2014] as

second move $\boxed{T} \circlearrowright \mathbf{Set}$

first move

$$\begin{tikzcd} \boxed{R \star T = URF} \circlearrowright \mathbf{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(T) \boxed{\circlearrowleft R} \arrow[l, bend left, "U"] \end{tikzcd}$$

alternating branching

Modalities for alternation

	Non-alternating	Alternating
Branching	$\begin{tikzcd} \mathbf{Set} \arrow[loop left, "T"] \end{tikzcd}$	$\begin{tikzcd} \mathbf{Set} \arrow[loop left, "R\star T=URF"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(T) \arrow[l, bend left, "U"] \arrow[loop right, "R"] \end{tikzcd}$
Modality	$\begin{tikzcd} \mathcal{D}^{\mathrm{op}} \arrow[d, bend left, "\mathcal{D}(-,A)"] \arrow[d, phantom, "\perp"] \\ \mathbf{Set} \arrow[u, bend left, "A^{(-)}"] \arrow[loop below] \end{tikzcd}$ $\tau\colon T \to \mathcal{D}(A^{(-)},A)$	$\begin{tikzcd} \mathcal{D}^{\mathrm{op}} \arrow[d, bend left, "{[-,\overline{\Omega}]_{\mathcal{D}}}"] \arrow[d, phantom, "\perp"] \\ \mathcal{EM}(T) \arrow[u, bend left, "{[-,\overline{\Omega}]_T}"] \arrow[loop below] \end{tikzcd}$ $\rho\colon R \to [[-,\overline{\Omega}]_T,\overline{\Omega}]_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{D}^{\mathrm{op}} \arrow[rr, bend left, "{[-,\overline{\Omega}]_{\mathcal{D}}}"] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(T) \arrow[ll, bend left, "{[-,\overline{\Omega}]_T}"] \\ & \mathcal{K}\ell(T) \arrow[ul, "\mathbb{P}^{\tau}"] \arrow[ur, "K"'] & \end{tikzcd}$$

Results

- Using these two modalities,

modalities for

2nd branching

$$\tau\colon T \to \mathcal{D}(A^{(-)}, A)$$

1st branching

$$\rho\colon R \to [[-, \overline{\Omega}]_T, \overline{\Omega}]_{\mathcal{D}}$$

we have a PT-semantics:

$$\mathbb{P}^{\tau,\rho}\colon \mathcal{K}\ell(R \star T) \longrightarrow \mathcal{D}^{\mathrm{op}}$$

- Its healthiness result is in the paper.

Summary of the alternting cases

- A dual adjunction between $\mathcal{D}$ and $\mathcal{EM}(T)$
  - a part of a state-and-effect triangle
- Our result naturally extends to the alternating cases
  - using the dual adjunction above

Future works

- Investigate **relative algebra**
  - especially its connection to **Lawvere theory**
- Extend result to enriched settings (cf. [Keimel 2015])