Varieties, Quasivarieties and Prevarieties: Completing the Picture

Wataru Hino & Ichiro Hasuo

The University of Tokyo

Outline

- Review 1: variety and quasivariety
- Review 2: orthogonality and prevariety
- New notion: sort-of-variety

New

characterization	Variety	Sort-of-variety	Quasivariety	Prevariety
logical	equations $\forall \vec{x}\ (s = t)$	∀∃!-formulas $\forall \vec{x}\ \exists! \vec{y}\ E$	implications $\forall \vec{x}\ (E \to s = t)$	preequations $\forall \vec{x}\ (E \to \exists! \vec{y}\ E')$
orthogonality	$FX \twoheadrightarrow A$	$FX \longrightarrow A$	$A \twoheadrightarrow B$	$A \longrightarrow B$
closure property	H, S, P	?	S, P, FC	A-pure S, P, FC

Review: variety and quasivariety

characterization	Variety	Sort-of-variety	Quasivariety	Prevariety
logical	equations $\forall \vec{x}\ (s = t)$	∀∃!-formulas $\forall \vec{x}\ \exists!\, \vec{y}\ E$	implications $\forall \vec{x}\ (E \to s = t)$	preequations $\forall \vec{x}\ (E \to \exists!\, \vec{y}\ E')$
orthogonality	$FX \twoheadrightarrow A$	$FX \longrightarrow A$	$A \twoheadrightarrow B$	$A \longrightarrow B$
closure property	H, S, P	?	S, P, FC	A-pure S, P, FC

Variety: definition

Def. (variety)

$V \subset \mathrm{Alg}\Sigma$ is a *variety*

$\Leftrightarrow {}^{\exists}E$: a set of equations

s.t. $V = \{A \in \mathrm{Alg}\Sigma \mid A \vDash E\}$

$\forall \vec{x}\ (s = t)$

signature-relevant definition

Example

- The class of groups is a variety for $\{\cdot, 1, {}^{-1}\}$, but not for $\{\cdot\}$
- Rings, lattices (for *appropriate* signatures)

Variety: HSP theorem

Thm. (Birkhoff)

$V \subset \mathrm{Alg}\Sigma$ is a variety

$\Longleftrightarrow V$ is closed under

- (H) homomorphic images
- (S) subalgebras
- (P) products

Variety can be characterized by *closure property*

Cor. The class of torsion-free abelian groups isn't a variety for $\{+, 0, -\}$

Quasivariety

Def. (quasivariety)

$V \subseteq \mathrm{Alg}\Sigma$ is a *quasivariety*

$\Leftrightarrow {}^{\exists}E$: a set of implications

s.t. $V = \{A \in \mathrm{Alg}\Sigma \mid A \vDash E\}$

$\forall \vec{x}\ (\wedge\, s_i = t_i \to s = t)$

Example Torsion-free abelian groups for $\{+, 0, -\}$

- *group-equations* & $\forall x\, (n \cdot x = 0 \to x = 0)$ for each n

Thm. $V \subset \mathrm{Alg}\Sigma$ is a quasivariety

$\Leftrightarrow$ V is closed under

(S) subalgebras, (P) products and (FC) filtered colimits

Review: orthogonality

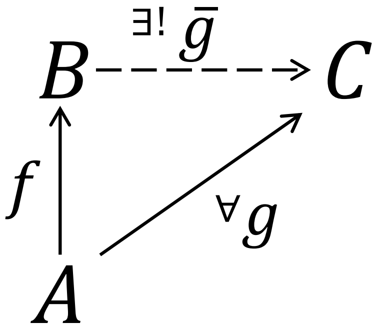
characterization	Variety	Sort-of-variety	Quasivariety	Prevariety
logical	equations $\forall\vec{x}\ (s = t)$	$\forall\exists!$-formulas $\forall\vec{x}\ \exists!\,\vec{y}\ E$	implications $\forall\vec{x}\ (E \to s = t)$	preequations $\forall\vec{x}\ (E \to \exists!\,\vec{y}\ E')$
orthogonality	$FX \twoheadrightarrow A$	$FX \longrightarrow A$	$A \twoheadrightarrow B$	$A \longrightarrow B$
closure property	H, S, P	?	S, P, FC	A-pure S, P, FC

Orthogonality [Freyd, Kelly 1972]

<u>Def.</u> (orthogonality)

$f : A \to B$ is *orthogonal* to C $\Longleftrightarrow$

$(f \perp C)$



<u>Example</u>

In the category of groups,

$$(\pi\colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}) \perp G \iff G \vDash \forall x\,(x^n = e)$$

$$(\iota\colon 2\mathbb{Z} \hookrightarrow \mathbb{Z}) \perp G \iff G \vDash \forall x \exists! y\,(y^2 = x)$$

Orthogonality

<u>Observation</u> FX, A, B: finitely presentable (FP)

- φ: equation $\longleftrightarrow$ $f\colon FX \twoheadrightarrow A$ (FX: free)
- φ: implication $\longleftrightarrow$ $f\colon A \twoheadrightarrow B$

Variety and quasivariety are characterized by orthogonality

→ What if we drop these conditions on morphisms: surjectivenss and free-domain?

Review: prevariety

characterization	Variety	Sort-of-variety	Quasivariety	Prevariety
logical	equations $\forall \vec{x}\ (s = t)$	∀∃!-formulas $\forall \vec{x}\ \exists! \vec{y}\ E$	implications $\forall \vec{x}\ (E \to s = t)$	preequations $\forall \vec{x}\ (E \to \exists! \vec{y}\ E')$
orthogonality	$FX \twoheadrightarrow A$	$FX \longrightarrow A$	$A \twoheadrightarrow B$	$A \longrightarrow B$
closure property	H, S, P	?	S, P, FC	A-pure S, P, FC

Prevariety [Adámek, Sousa 2004]

<u>Def.</u> (prevariety a.k.a. *ω-orthogonality class*)

$V \subset \mathrm{Alg}\Sigma$ is a *prevariety*

$\Longleftrightarrow$ $^{\exists}M$: a set of morphisms (between FP algebra)

s.t. $V = \{A \in \mathrm{Alg}\Sigma \mid M \perp A\}$

correspond to *pre-equation*
$\forall \vec{x}\ (E \to \exists!\, \vec{y}\, E')$

- It also can be characterized by closure property

2 axes for variety-like notions

	Free-domain	Arbitrary
Surjective	Variety: $FX \twoheadrightarrow A$	Quasi-variety: $A \twoheadrightarrow B$
Arbitrary	Sort-of-variety: $FX \rightarrow A$	Prevariety: $A \rightarrow B$

characterization	Variety	Sort-of-variety	Quasivariety	Prevariety
logical	equations $\forall\vec{x}\ (s = t)$	∀∃!-formulas $\forall\vec{x}\ \exists!\vec{y}\ E$	implications $\forall\vec{x}\ (E \rightarrow s = t)$	preequations $\forall\vec{x}\ (E \rightarrow \exists!\vec{y}\ E')$
orthogonality	$FX \twoheadrightarrow A$	$FX \longrightarrow A$	$A \twoheadrightarrow B$	$A \longrightarrow B$
closure property	H, S, P	?	S, P, FC	A-pure S, P, FC

New notion: sort-of-variety

characterization	Variety	Sort-of-variety	Quasivariety	Prevariety
logical	equations $\forall \vec{x}\ (s = t)$	$\forall\exists!$-formulas $\forall \vec{x}\ \exists!\, \vec{y}\ E$	implications $\forall \vec{x}\ (E \to s = t)$	preequations $\forall \vec{x}\ (E \to \exists!\, \vec{y}\ E')$
orthogonality	$FX \twoheadrightarrow A$	$FX \longrightarrow A$	$A \twoheadrightarrow B$	$A \longrightarrow B$
closure property	H, S, P	?	S, P, FC	A-pure S, P, FC

Sort-of-variety: definition

orthogonality to
$f : FX \to A$

Def. (sort-of-variety)

$V \subset \mathrm{Alg}\Sigma$ is a *sort-of-variety*

$\forall \vec{x}\ \exists!\, \vec{y}\, E$

$\Leftrightarrow {}^{\exists}E$: a set of *∀∃!-formulas*

s.t. $V = \{A \in \mathrm{Alg}\Sigma \mid A \vDash E\}$

Remark

- ∀∃!-formula defines "extra" operations on algebras
  - Σ-morphisms must preserve them

Sort-of-variety: example

<u>Example</u>

The class of groups is a sort-of-variety for $\{\cdot\}$

- neither a variety nor a quasivariety
- associativity, $\forall xy \exists! z\, (x \cdot z = y), \forall xy \exists! z\, (z \cdot x = y)$ and $\exists! e\, (e^2 = e)$
- They define 3 extra operations $\{\, \backslash,\ / \,, e\}$
    - which satisfy $x \cdot (x \backslash y) = y$ etc.
    - e is unit and $x^{-1} = x \backslash e$
- $\{\cdot\}$-morphisms preserve unit and multiplication

Relation between other variety-like notions

<u>Thm.</u>

A sort-of-variety for Σ is isomorphic to a quasivariety for an extended signature Σ' (by the forgetful functor).

	Free-domain	**Arbitrary**
Surjective	Variety	Quasi-variety
Arbitrary	Sort-of-variety	Prevariety

Expanding Σ

Converse?

<u>Observation</u>

- The quasivariety of *torsion-free abelian groups* and that of *positive monoids* are a sort-of-variety.
- The class of *left-cancellative monoids* is quasivariety, but doesn't seem to be a sort-of-variety.

	Free-domain	**Arbitrary**
Surjective	Variety	Quasi-variety
Arbitrary	Sort-of-variety	?? Prevariety

Summary

- Reviewed variety, quasivariety and prevariety
- Introduced a new variety-like notion: *sort-of-variety*

Future work

- Find the closure property for sort-of-variety
- Exploit its relation between other variety-like notions